An International Journal
computers & mathematics
with applications

PERGAMON Computers and Mathematics with Applications 38 (1999) 71–78
www.elsevier.nl/locate/camwa

A Unified Presentation of Certain Subclasses of Prestarlike Functions with Negative Coefficients

R. K. Raina
Department of Mathematics, College of Technology and Agricultural Engineering
Rajasthan Agricultural University, Udaipur 313001, Rajasthan, India
raina_rk@yahoo.com

H. M. Srivastava
Department of Mathematics and Statistics, University of Victoria
Victoria, British Columbia V8W 3P4, Canada
harimsri@math.uvic.ca

(Received April 1999; accepted May 1999)

Abstract—The main object of this paper is to introduce and investigate various properties and characteristics of a unified class $\mathcal{P}(\alpha, \beta, \sigma)$, of prestarlike functions with negative coefficients. The results presented here involve distortion inequalities and modified Hadamard products (or convolution) of functions belonging to the class $\mathcal{P}(\alpha, \beta, \sigma)$. Growth and distortion theorems involving fractional integrals and fractional derivatives are also considered. Relevant connections of some of these results with those given in earlier works are briefly pointed out.

Keywords—Prestarlike functions, Analytic functions, Univalent functions, Starlike functions, Hadamard product (or convolution), Cauchy-Schwarz inequality, Growth and distortion theorems, Fractional integrals and fractional derivatives, Riemann-Liouville operators.

1. INTRODUCTION

We denote by $\mathcal{S}$ the class of (*normalized*) functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* and *univalent* in the open unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

The present investigation was supported (in part) by the National Board for Higher Mathematics (Department of Atomic Energy) of the Government of India under Grant 26/6/93-G and (in part) by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353.

PII: S0898-1221(99)00286-2

The familiar Hadamard product (or convolution) of two functions $f(z)$ given by (1.1), and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.2}$$

is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.3}$$

Following the work of Sheil-Small *et al.* [1], we define the subclass $\mathcal{R}(\alpha, \beta)$ of $\mathcal{S}$ consisting of *α-prestarlike functions of order β* by

$$\mathcal{R}(\alpha,\beta) := \{f \in \mathcal{S} : (f * s_\alpha)(z) \in \mathcal{S}^*(\beta),\ (0 \le \alpha < 1; 0 \le \beta < 1)\}, \tag{1.4}$$

where $\mathcal{S}^*(\beta)$ denotes the class of *starlike functions of order* β $(0 \le \beta < 1)$ and $s_\alpha(z)$ is the well-known extremal function for $\mathcal{S}^*(\alpha)$ given by (cf., e.g., [2,3])

$$\begin{aligned} s_\alpha(z) &:= \frac{z}{(1-z)^{2(1-\alpha)}}, \qquad (z \in \mathcal{U};\ 0 \le \alpha < 1) \\ &= z + \sum_{n=2}^{\infty} c_n(\alpha) z^n \end{aligned} \tag{1.5}$$

with

$$c_n(\alpha) := \frac{\prod\limits_{k=2}^{n} (k - 2\alpha)}{(n-1)!}, \qquad (n \in \mathbb{N} \setminus \{1\};\ \mathbb{N} := \{1, 2, 3, \dots\}). \tag{1.6}$$

We also define the subclass $\mathcal{C}(\alpha, \beta)$ of $\mathcal{S}$ by

$$\mathcal{C}(\alpha,\beta) := \{f \in \mathcal{S} : zf'(z) \in \mathcal{R}(\alpha,\beta)\}. \tag{1.7}$$

Let $\mathcal{T}$ denote the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad (a_n \ge 0). \tag{1.8}$$

We denote by $\mathcal{R}[\alpha, \beta]$ and $\mathcal{C}[\alpha, \beta]$ the classes obtained by taking intersections, respectively, of the classes $\mathcal{R}(\alpha, \beta)$ and $\mathcal{C}(\alpha, \beta)$ with the class $\mathcal{T}$. Thus, we have (cf. [4, p. 55])

$$\mathcal{R}[\alpha,\beta] := \mathcal{R}(\alpha,\beta) \cap \mathcal{T} \tag{1.9}$$

and

$$\mathcal{C}[\alpha,\beta] := \mathcal{C}(\alpha,\beta) \cap \mathcal{T}. \tag{1.10}$$

The following known results for the classes $\mathcal{R}[\alpha, \beta]$ and $\mathcal{C}[\alpha, \beta]$ will be required in our present investigation.

LEMMA 1. *(See [5].) Let $f(z)$ be defined by (1.8). Then $f(z) \in \mathcal{R}[\alpha, \beta]$ if and only if*

$$\sum_{n=2}^{\infty} (n - \beta) c_n(\alpha) a_n \le 1 - \beta. \tag{1.11}$$

The result is sharp.

LEMMA 2. *(See [6].) Let $f(z)$ be defined by (1.8). Then $f(z) \in \mathcal{C}[\alpha,\beta]$ if and only if*

$$\sum_{n=2}^{\infty} n(n-\beta)c_n(\alpha)a_n \le 1-\beta. \tag{1.12}$$

The result is sharp.

In view of Lemma 1 and Lemma 2, we find it to be worthwhile to present here a unified study of the classes $\mathcal{R}[\alpha,\beta]$ and $\mathcal{C}[\alpha,\beta]$ by introducing a new class $\mathcal{P}(\alpha,\beta,\sigma)$. Indeed, we say that a function $f(z)$ defined by (1.8) belongs to the class $\mathcal{P}(\alpha,\beta,\sigma)$, if and only if

$$\sum_{n=2}^{\infty} \left[\frac{(n-\beta)(1-\sigma+\sigma n)}{1-\beta}\right] c_n(\alpha)a_n \le 1, \tag{1.13}$$

$$(0 \le \alpha < 1;\ 0 \le \beta < 1;\ 0 \le \sigma \le 1).$$

Clearly, we have

$$\mathcal{P}(\alpha,\beta,\sigma) = (1-\sigma)\mathcal{R}[\alpha,\beta] + \sigma\mathcal{C}[\alpha,\beta], \qquad (0 \le \sigma \le 1), \tag{1.14}$$

so that

$$\mathcal{P}(\alpha,\beta,0) = \mathcal{R}[\alpha,\beta] \quad \text{and} \quad \mathcal{P}(\alpha,\beta,1) = \mathcal{C}[\alpha,\beta]. \tag{1.15}$$

The main purpose of the present paper is to investigate various properties and chracteristics of the general class $\mathcal{P}(\alpha,\beta,\sigma)$. We also indicate relevant connections of some of our results with those given in earlier works on the subject of investigation here.

2. A SET OF DISTORTION INEQUALITIES

We first establish the following distortion inequalities for functions belonging to the class $\mathcal{P}(\alpha,\beta,\sigma)$.

THEOREM 1. *If a function $f(z)$ defined by (1.8) is in the class $\mathcal{P}(\alpha,\beta,\sigma)$, then*

$$|z| - \frac{1-\beta}{2(2-\beta)(1+\sigma)(1-\alpha)}|z|^2 \le |f(z)| \le |z| + \frac{1-\beta}{2(2-\beta)(1+\sigma)(1-\alpha)}|z|^2 \tag{2.1}$$

and

$$1 - \frac{1-\beta}{(2-\beta)(1+\sigma)(1-\alpha)}|z| \le |f'(z)| \le 1 + \frac{1-\beta}{(2-\beta)(1+\sigma)(1-\alpha)}|z| \tag{2.2}$$

$$\left(z \in \mathcal{U};\ 0 \le \alpha \le \frac{1}{2};\ 0 \le \beta < 1;\ 0 \le \sigma \le 1\right).$$

PROOF. Observing that $c_n(\alpha)$ defined by (1.6) is nondecreasing for $0 \le \alpha \le 1/2$, we find from (1.13) that

$$\sum_{n=2}^{\infty} a_n \le \frac{1-\beta}{2(2-\beta)(1+\sigma)(1-\alpha)}. \tag{2.3}$$

Using (1.8) and (2.3), we readily have (for $z \in \mathcal{U}$)

$$|f(z)| \ge |z| - |z|^2 \sum_{n=2}^{\infty} a_n \ge |z| - \frac{1-\beta}{2(2-\beta)(1+\sigma)(1-\alpha)}|z|^2 \tag{2.4}$$

and

$$|f(z)| \le |z| + |z|^2 \sum_{n=2}^{\infty} a_n \le |z| + \frac{1-\beta}{2(2-\beta)(1+\sigma)(1-\alpha)}|z|^2, \tag{2.5}$$

which prove the assertion (2.1) of Theorem 1.

Also, from (1.8), we find for $z \in \mathcal{U}$ that

$$|f'(z)| \geq 1 - |z| \sum_{n=1}^{\infty} n\, a_n \geq 1 - \frac{1-\beta}{(2-\beta)(1+\sigma)(1-\alpha)}\, |z| \tag{2.6}$$

and

$$|f'(z)| \leq 1 + |z| \sum_{n=2}^{\infty} n\, a_n \leq 1 + \frac{1-\beta}{(2-\beta)(1+\sigma)(1-\alpha)}\, |z|\,, \tag{2.7}$$

which prove the assertion (2.2) of Theorem 1.

Finally, since each of the results (2.1) and (2.2) is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{1-\beta}{2(2-\beta)(1+\sigma)(1-\alpha)} z^2, \tag{2.8}$$

we complete the proof of Theorem 1.

3. PROPERTIES INVOLVING HADAMARD PRODUCTS (OR CONVOLUTION)

For the functions $f_1(z)$ and $f_2(z)$ defined by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \qquad (a_{n,j} \geq 0;\ j = 1,2)\,, \tag{3.1}$$

we define here the *modified* Hadamard product (or convolution) by (cf. equation (1.3))

$$(f_1 \bullet f_2)(z) := z - \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n. \tag{3.2}$$

We now prove the following.

THEOREM 2. *Let each of the functions $f_j(z)\,(j = 1,2)$ be in the class $\mathcal{P}(\alpha,\beta,\sigma)$. Then the modified Hadamard product $(f_1 \bullet f_2)(z)$ belongs to the class $\mathcal{P}(\alpha,\rho,\sigma)$, where*

$$\rho := \frac{2(2-\beta)^2(1+\sigma)(1-\alpha) - 2(1-\beta)^2}{2(2-\beta)^2(1+\sigma)(1-\alpha) - (1-\beta)^2}, \tag{3.3}$$

$$\left(0 \leq \alpha \leq \frac{1}{2};\ 0 \leq \beta < 1;\ 0 \leq \sigma \leq 1\right).$$

PROOF. We need to find the largest ρ such that

$$\sum_{n=2}^{\infty} \left[\frac{(n-\rho)(1-\sigma+\sigma n)}{1-\rho}\right] c_n(\alpha) a_{n,1} a_{n,2} \leq 1. \tag{3.4}$$

By hypothesis, $f_j(z) \in \mathcal{P}(\alpha,\beta,\sigma)\,(j = 1,2)$, so definition (1.13) yields

$$\sum_{n=2}^{\infty} \left[\frac{(n-\beta)(1-\sigma+\sigma n)}{1-\beta}\right] c_n(\alpha) a_{n,j} \leq 1, \qquad (j = 1,2). \tag{3.5}$$

Applying the Cauchy-Schwarz inequality, we get

$$\sum_{n=2}^{\infty} \left[\frac{(n-\beta)(1-\sigma+\sigma n)}{1-\beta}\right] c_n(\alpha) \sqrt{a_{n,1} a_{n,2}} \leq 1. \tag{3.6}$$

Thus, it is sufficient to show that

$$\begin{aligned}&\sum_{n=2}^{\infty}\left[\frac{(n-\rho)(1-\sigma+\sigma n)}{1-\rho}\right]c_n(\alpha)a_{n,1}a_{n,2}\\&\leq\sum_{n=2}^{\infty}\left[\frac{(n-\beta)\,(1-\sigma+\sigma n)}{1-\beta}\right]c_n(\alpha)\,\sqrt{a_{n,1}a_{n,2}},\end{aligned}\tag{3.7}$$

that is, that

$$\sqrt{a_{n,1}a_{n,2}}\leq\frac{(n-\beta)(1-\rho)}{(n-\rho)(1-\beta)},\qquad(n\geq 2).\tag{3.8}$$

Inequality (3.6) implies that

$$\sqrt{a_{n,1}a_{n,2}}\leq\frac{1-\beta}{(n-\beta)(1-\sigma+\sigma n)c_n(\alpha)},\qquad(n\geq 2).\tag{3.9}$$

It, therefore, suffices to prove that

$$\frac{1-\beta}{(n-\beta)(1-\sigma+\sigma n)c_n(\alpha)}\leq\frac{(n-\beta)(1-\rho)}{(n-\rho)(1-\beta)},\qquad(n\geq 2),\tag{3.10}$$

which will follow if

$$\rho\leq\frac{(n-\beta)^2(1-\sigma+\sigma n)c_n(\alpha)-n(1-\beta)^2}{(n-\beta)^2(1-\sigma+\sigma n)c_n(\alpha)-(1-\beta)^2}=:\Theta(n),\qquad(n\geq 2).\tag{3.11}$$

Putting

$$\Delta(n)=(n-\beta)^2(1-\sigma+\sigma n)c_n(\alpha),\qquad(n\geq 2),\tag{3.12}$$

we readily observe that $\Delta(n)>0$ $(n\geq 2)$ and

$$\Theta'(n)=\frac{(1-\beta)^2}{[\Delta(n)-(1-\beta)^2]^2}\left[(n-1)\Delta'(n)-\Delta(n)+(1-\beta)^2\right],\qquad(n\geq 2).\tag{3.13}$$

A simple calculation from (3.12) shows that

$$\begin{aligned}\Delta'(n)&=2(n-\beta)^{-1}\Delta(n)+\sigma(1-\sigma+\sigma n)^{-1}\Delta(n)\\&\quad+\Delta(n)\left\{\psi(n+1-2\alpha)-\psi(n)\right\},\end{aligned}\tag{3.14}$$

where $\psi(z)$ denotes the Psi (or Digamma) function defined by

$$\psi(z):=\frac{d}{dz}\left\{\log\Gamma(z)\right\}=\frac{\Gamma'(z)}{\Gamma(z)}\tag{3.15}$$

in terms of the Gamma function. Now, multiplying each member of (3.14) by $(n-1)/\Delta(n)$, we have

$$\begin{aligned}\frac{(n-1)\Delta'(n)}{\Delta(n)}&=\frac{2(n-1)}{n-\beta}+\frac{\sigma(n-1)}{1-\sigma+\sigma n}+(n-1)\left\{\psi(n+1-2\alpha)-\psi(n)\right\}\\&>(n-1)\left\{\psi(n+1-2\alpha)-\psi(n)\right\}\\&>0,\qquad\left(n\geq 2;\ 0\leq\alpha\leq\frac{1}{2}\right),\end{aligned}$$

which, in conjunction with (3.13), shows that $\Phi(n)$ is an increasing function of n $(n\geq 2)$ when $0\leq\alpha\leq 1/2$. Hence, we conclude from (3.11) that

$$\rho\leq\Theta(2)=\frac{2(2-\beta)^2(1+\sigma)(1-\alpha)-2(1-\beta)^2}{2(2-\beta)^2(1+\sigma)(1-\alpha)-(1-\beta)^2},\tag{3.16}$$

and the proof of the Theorem 2 is completed by observing that the result of Theorem 2 is sharp for the functions $f_j(z)$ $(j=1,2)$ given by (2.8).

THEOREM 3. *Let each of the functions $f_j(z)$ $(j = 1, 2)$ defined by (3.1) be in the class $\mathcal{P}(\alpha, \beta, \sigma)$. Then the function $h(z)$ defined by*

$$h(z) := z - \sum_{n=2}^{\infty} \left(a_{n,1}^2 + a_{n,2}^2\right) z^n \tag{3.17}$$

belongs to the class $\mathcal{P}$ (α, δ, σ), where

$$\delta := \frac{(2-\beta)^2(1+\sigma)(1-\alpha) - 2(1-\beta)^2}{(2-\beta)^2(1+\sigma)(1-\alpha) - (1-\beta)^2}, \tag{3.18}$$

$$\left(0 \le \alpha \le \frac{1}{2};\ 0 \le \beta < 1;\ 0 \le \sigma \le 1\right).$$

PROOF. In view of the hypothesis of Theorem 3, it is easily seen that

$$\sum_{n=2}^{\infty} \left[\frac{(n-\beta)^2(1-\sigma+\sigma n)^2}{(1-\beta)^2}\right] \{c_n(\alpha)\}^2 a_{n,j}^2$$
$$\le \left(\sum_{n=2}^{\infty} \left[\frac{(n-\beta)(1-\sigma+\sigma n)}{(1-\beta)}\right] c_n(\alpha) a_{n,j}\right)^2 \le 1, \qquad (j = 1, 2). \tag{3.19}$$

Therefore,

$$\sum_{n=2}^{\infty} \left[\frac{(n-\beta)^2(1-\sigma+\sigma n)^2}{(1-\beta)^2}\right] \{c_n(\alpha)\}^2 \left(a_{n,1}^2 + a_{n,2}^2\right) \le 2. \tag{3.20}$$

It is sufficient to find the largest δ such that

$$\frac{(n-\delta)(1-\sigma+\sigma n)c_n(\alpha)}{1-\delta} \le \frac{(n-\beta)^2(1-\sigma+\sigma n)^2 \{c_n(\alpha)\}^2}{2(1-\beta)^2},$$

which follows when

$$\delta \le \frac{(n-\beta)^2(1-\sigma+\sigma n)c_n(\alpha) - 2n(1-\beta)^2}{(n-\beta)^2(1-\sigma+\sigma n)c_n(\alpha) - 2(1-\beta)^2} := \Phi(n), \qquad (n \ge 2). \tag{3.21}$$

The remainder of our proof would run parallel to that of Theorem 2, which we have already detailed above.

4. GROWTH AND DISTORTION THEOREMS INVOLVING FRACTIONAL CALCULUS OPERATORS

We begin by recalling the fractional integral operator $I_{0,z}^{\lambda,\mu,\eta}$ as follows.

DEFINITION 1. *(See [7].) Let $\lambda \in \mathbb{R}_+$ and $\mu, \eta \in \mathbb{R}$. Then, in terms of the familiar (Gaùss's) hypergeometric function ${}_2F_1$, the fractional integral operator $I_{0,z}^{\lambda,\mu,\eta}$ is defined by*

$$I_{0,z}^{\lambda,\mu,\eta} f(z) := \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1}\, {}_2F_1\left(\lambda+\mu, -\eta;\ \lambda;\ 1-\frac{t}{z}\right) f(t)\, dt, \tag{4.1}$$

where the function $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin, with the order

$$f(z) = O\left(|z|^{\kappa}\right), \qquad (z \to 0) \tag{4.2}$$

for

$$\kappa > \max\{0, \mu - \eta\} - 1, \tag{4.3}$$

and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $z - t \in \mathbb{R}_+$.

Next, under the same constraints as in Definition 1 above, an extended definition of the fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ is given by

$$J_{0,z}^{\lambda,\mu,\eta} f(z) := \frac{d^n}{dz^n}\left(\frac{z^{\lambda-\mu}}{\Gamma(n-\lambda)}\int_o^z (z-t)^{n-\lambda-1}\,{}_2F_1\left(\mu-\lambda, n-\eta;\; n-\lambda;\; 1-\frac{t}{z}\right) f(t)\,dt\right), \tag{4.4}$$

$$(n-1 \le \lambda < n;\; n \in \mathbb{N};\; \mu,\eta \in \mathbb{R}).$$

The fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ $(0 \le \lambda < 1)$, studied recently in [8,9], follows from (4.4) when $n = 1$. The fractional calculus operators defined by (4.1) and (4.4) include the Riemann-Liouville operators as their particular cases (cf. [10–12]):

$$I_{0,z}^{\lambda,-\lambda,\eta} f(z) = {}_0D_z^{-\lambda} f(z), \qquad (\lambda > 0) \tag{4.5}$$

and

$$J_{0,z}^{\lambda,\lambda,\eta} f(z) = {}_0D_z^{\lambda} f(z), \qquad (\lambda \ge 0). \tag{4.6}$$

We now prove the following growth and distortion theorems involving the fractional calculus operators defined by (4.1) and (4.4).

THEOREM 4. *Let $\lambda \in \mathbb{R}_+$ and $\mu,\eta \in \mathbb{R}$ such that*

$$\mu < 2, \quad \eta > \max\{-\lambda,\mu\} - 2, \quad \text{and} \quad \mu(\lambda+\eta) \le 3\lambda.$$

If $f(z) \in \mathcal{P}(\alpha,\beta,\sigma)$ for $0 \le \alpha \le 1/2$, $0 \le \beta < 1$, and $0 \le \sigma \le 1$, then

$$\begin{aligned}&\left|I_{0,z}^{\lambda,\mu,\eta} f(z)\right| \\ &\ge \frac{\Gamma(2-\mu+\eta)}{\Gamma(2-\mu)\Gamma(2+\lambda+\eta)}|z|^{1-\mu}\left\{1 - \frac{(1-\beta)(2-\mu+\eta)}{(2-\beta)(1+\sigma)(1-\alpha)(2-\mu)(2+\lambda+\eta)}|z|\right\}\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}&\left|I_{0,z}^{\lambda,\mu,\eta} f(z)\right| \\ &\le \frac{\Gamma(2-\mu+\eta)}{\Gamma(2-\mu)\Gamma(2+\lambda+\eta)}|z|^{1-\mu}\left\{1 + \frac{(1-\beta)(2-\mu+\eta)}{(2-\beta)(1+\sigma)(1-\alpha)(2-\mu)(2+\lambda+\eta)}|z|\right\}\end{aligned} \tag{4.8}$$

$$(z \in \mathcal{U},\ \text{if } \mu \le 1;\ z \in \mathcal{U}\setminus\{0\}\ \text{if } \mu > 1).$$

The equalities in (4.7) and (4.8) are attained by the function $f(z)$ given by (2.8).

PROOF. Since $f(z) \in \mathcal{P}(\alpha,\beta,\sigma)$ for $0 \le \alpha \le 1/2$, $0 \le \beta < 1$, and $0 \le \sigma \le 1$, it follows from (1.13) that

$$\begin{aligned}&\frac{2(2-\beta)(1+\sigma)(1-\alpha)}{1-\beta}\sum_{n=2}^{\infty} a_n \\ &\le \sum_{n=2}^{\infty}\left[\frac{(n-\beta)(1-\sigma+\sigma n)}{1-\beta}\right] c_n(\alpha) a_n \le 1,\end{aligned} \tag{4.9}$$

which gives

$$\sum_{n=2}^{\infty} a_n \le \frac{1-\beta}{2(2-\beta)(1+\sigma)(1-\alpha)}. \tag{4.10}$$

From (1.12), (4.1), and the known formula [7, p. 415, Lemma 3], we obtain

$$\left|I_{0,z}^{\lambda,\mu,\eta} f(z)\right| \ge \frac{\Gamma(2-\mu+\eta)}{\Gamma(2-\mu)\Gamma(2+\lambda+\eta)}\left\{1 - |z|\sum_{n=2}^{\infty}\Psi(n)a_n\right\}, \tag{4.11}$$

where

$$\Psi(n) := \frac{(2)_{n-1}(2-\mu+\eta)_{n-1}}{(2-\mu)_{n-1}(2+\lambda+\eta)_{n-1}}, \qquad (n \geq 2), \tag{4.12}$$

with $(\lambda)_n := \Gamma(\lambda+n)/\Gamma(\lambda)$. We observe that the function $\Psi(n)$ is a nonincreasing function of n, under the hypotheses of Theorem 4, and the desired inequality (4.7) is easily obtained on using (4.10) to (4.12).

Assertion (4.8) can be proved in a similar manner.

REMARK 1. In view of the relationships (1.15) and (4.5), a special case of Theorem 4 when $\mu = -\lambda$ would correspond to Theorem 1 and Theorem 2 of [4] for $\sigma = 0$ and $\sigma = 1$, respectively.

REMARK 2. In its special cases when $\sigma = 0$ and $\sigma = 1$, Theorem 4 also yields Theorem 5 and Theorem 6, respectively, of [4].

The proof of the following growth and distortion inequalities for the fractional derivative of $f(z)$ belonging to the class $\mathcal{P}(\alpha, \beta, \sigma)$ would run parallel to that of Theorem 4.

THEOREM 5. *Let $\lambda \in \mathbb{R}_+$ and $\mu, \eta \in \mathbb{R}$ such that*

$$\mu < 2, \quad \eta > \max\{\lambda, \mu\} - 2, \quad \text{and} \quad \mu(\lambda - \eta) \geq 3\lambda.$$

If $f(z) \in \mathcal{P}(\alpha, \beta, \sigma)$ for $0 \leq \alpha \leq 1/2$, $0 \leq \beta < 1$, and $0 \leq \sigma \leq 1$, then

$$\left| J_{0,z}^{\lambda,\mu,\eta} f(z) \right| \geq \frac{\Gamma(2-\mu+\eta)}{\Gamma(2-\mu)\Gamma(2-\lambda+\eta)} |z|^{1-\mu} \left\{ 1 - \frac{(1-\beta)(2-\mu+\eta)}{(2-\beta)(1+\sigma)(2-\mu)(2-\lambda+\eta)} |z| \right\} \tag{4.13}$$

and

$$\left| J_{0,z}^{\lambda,\mu,\eta} f(z) \right| \leq \frac{\Gamma(2-\mu+\eta)}{\Gamma(2-\mu)\Gamma(2-\lambda+\eta)} |z|^{1-\mu} \left\{ 1 + \frac{(1-\beta)(2-\mu+\eta)}{(2-\beta)(1+\sigma)(2-\mu)(2-\lambda+\eta)} |z| \right\} \tag{4.14}$$

$$(z \in \mathcal{U}, \text{ if } \mu \leq 1;\ z \in \mathcal{U} \setminus \{0\}, \text{ if } \mu > 1).$$

The equalities in (4.13) and (4.14) are attained by the function $f(z)$ given by (2.8).

REMARK 3. For $\mu = \lambda$, Theorem 5 corresponds to Theorem 3 and Theorem 4 of [4] when $\sigma = 0$ and $\sigma = 1$, respectively. These special cases of Theorem 5 hold true for rather less restrictive condition for λ (and thus, provide slightly improved versions of the growth and distortion inequalities involving ${}_0D_z^\lambda$, which were established in [4] for $0 \leq \lambda < 1$).

REFERENCES

1. T. Sheil-Small, H. Silverman and E. Silvia, Convolution multipliers and starlike functions, *J. Analyse Math.* **41**, 181–192, (1982).
2. P.L. Duren, *Univalent Functions*, Grundlehrender Mathematischen Wissenschaften, Volume 259, Springer-Verlag, New York, (1983).
3. H.M. Srivastava and S. Owa, Editors, *Current Topics in Analytic Function Theory*, World Scientific, Singapore, (1992).
4. H.M. Srivastava and M.K. Aouf, Some applications of fractional calculus operators to certain subclasses of prestarlike functions with negative coefficient, *Computers Math. Applic.* **30** (1), 53–61, (1995).
5. H. Silverman and E. Silvia, Subclasses of prestarlike functions, *Math. Japon.* **29**, 929–935, (1984).
6. S. Owa and B.A. Uralegaddi, A class of functions α-prestarlike of order β, *Bull. Korean Math. Soc.* **21**, 77–85, (1984).
7. H.M. Srivastava, M. Saigo and S. Owa, A class of distortion theorems involving certain operators of fractional calculus, *J. Math. Anal. Appl.* **131**, 412–420, (1988).
8. R.K. Raina, On certain classes of analytic functions and applications to fractional calculus operators, *Integral Transform. Spec. Funct.* **5**, 247–260, (1997).
9. R.K. Raina and H.M. Srivastava, A certain subclass of analytic functions associated with operators of fractional calculus, *Computers Math. Applic.* **2** (7), 13–19, (1996).
10. K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley-Interscience, New York, (1993).
11. S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach Science, Reading, PA, (1993).
12. H.M. Srivastava and S. Owa, Editors, *Univalent Functions, Fractional Calculus, and Their Applications*, Halsted Press, Chichester, (1989).